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Some statistical properties of the even and the odd negative binomial states

Amitabh Joshi†§ and A-S F Obada‡

† Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK

‡ Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt

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Abstract. New states of electromagnetic field, i.e. even and odd negative binomial states are introduced here. These states interpolate between the even (odd) coherent states and the even (odd) quasi-thermal states. Various statistical properties of these states such as the mean photon number, second-order coherence function, normal squeezing are calculated. The Wigner function for these states are also discussed. These results may be useful for carrying out systematic study of the non-classical properties of 'Schrödinger cat states' as one moves from even (odd) coherent state to even (odd) quasi-thermal state.

1. Introduction

In the past few decades many electromagnetic field states, which are in general a superposition of Fock states, have been introduced in quantum optics. Most of these states are characterized by some discrete photon number distribution and thus play a significant role in describing the statistical properties of the radiation field. For example, fields described by the coherent state and the thermal state are known to lead to Poisson and Bose–Einstein distributions respectively for the number of photons [1]. Binomial, as well as negative binomial, states of the field are introduced and their statistical properties and interaction with the matter are also reported [2–5]. The interesting property of a binomial (negative binomial) state is that under two different limiting conditions it reduces to either a Fock state (quasi-thermal state) or a coherent state [2–5]. The special case of a negative binomial state with the removal of the $n = 0$ term, known as the logarithmic state, has been investigated and it is found to exhibit some non-classical characteristics and its interaction with matter is considered [6]. Also, the generalized geometric state [7] has been studied and this state interpolates between a Fock state and a thermal (non-pure) state. Recently, the superposition of two coherent states is investigated with the aim of understanding the role of quantum interference between coherent states and consequently to generate the states whose properties are different from an ordinary coherent state [8–12]. The possibility of generating such a superposition of the coherent states in experiments has been proposed by many authors. In fact, Yurke and Stoler [13] have shown that a coherent state propagating through an amplitude dispersive medium, under certain specified conditions of parameters can evolve into a superposition of two coherent states 180° out of phase. Another proposal has been made for generating the superposition $|\alpha\rangle \pm |-\alpha\rangle$ (i.e. even and odd coherent states) in

§ Permanent address: Laser and Plasma Technology Division, Bhabha Atomic Research Centre, Trombay, Bombay 400 085, India. E-mail address: Amitabh.Joshi@umist.ac.uk cavity QED by Brune *et al* [14] and Gea-Banocholche [15]. Generation of even and odd coherent states based on competition between parametric amplification and the incoherent losses from two-photon resonant absorption has also been proposed [12]. Very recently, one of us has proposed a new kind of superposition of binomial states which has been named as the even binomial state [16]. Some of the properties of this state along with some possibilities of its generation in quantum optical processes have also been discussed [16]. In this work we would discuss yet another class of the state(s) to be called by the even (odd) negative binomial state, which in fact, interpolates between the even (odd) coherent state and the even (odd) quasi-thermal state. The ordinary negative binomial state is always super-Poissonian no matter what parameters one chooses. However, if one considers a 'Schrödinger cat'-like superposition of two negative binomial states then, as we will see later, the super-Poissonian character is flipped over to a sub-Poissonian character for a certain range of parameters. Since the negative binomial state is 'intermediate' between a coherent state and a thermal-like (pure) state, by varying its parameters one can move systematically from a coherent-state character to a thermal-like state character. The 'Schrödinger cat'-like superposition of two coherent states has been well studied in the literature so it would be very interesting to study the superposition of such states which on the one hand represent the even (odd) coherent state and on the other extreme represent the even (odd) thermal state. Thus how the effect of quantum interference (due to 'Schrödinger cat'-like superposition of states) gets modified as one moves from the coherent state character to a thermal-like state character could be reasoned as the motivation behind this study. Systems exhibiting quantum coherence between parts differing by some macroscopic physical parameters has been given the generic name of 'Schrödinger cats'. The study of 'decoherence' in such a system is extremely interesting since it provides a testing ground for the ideas which are at the heart of the measurement theory in quantum mechanics. The rest of the paper is organized as follows. In section 2 we introduce the even and odd negative binomial state and calculate the mean photon number and second-order coherence function for them. Section 3 is devoted to the study of normal squeezing of these states. The Wigner function related to these states is calculated in section 4. Finally, some possibilities of generating these states and concluding remarks are given in section 5.

2. The even and the odd negative binomial states

The negative binomial state is defined in the literature [4, 5]:

$$|w, q\rangle = \sum_{n=0}^{\infty} \left[\frac{(n+w)!}{n!w!} q^{2n} (1-|q|^2)^{w+1} \right]^{1/2} |n\rangle \tag{1}$$

where $w \geqslant 0$ (i.e., w is any real positive number in general [4]); $0 \leqslant |q|^2 < 1$. The mean and the variance of photon number distribution corresponding to the state (1) is given by

$$\langle n\rangle = (w+1)\frac{|q|^2}{1-|q|^2} \qquad \langle n^2\rangle - \langle n\rangle^2 = (w+1)\frac{|q|^2}{(1-|q|^2)^2}. \tag{2}$$

The state (1) reduces to a coherent state in the limiting condition $w \to \infty$, $|q|^2 \to 0$ such that $(w+1)|q|^2/(1-|q|^2) \equiv \langle n\rangle$ is kept constant. However, in another limiting condition, i.e. $w = 0$, the state (1) represents a quasi-thermal state which is a pure state having photon number distribution identical to the usual thermal (chaotic) state (mixed state) with the same mean and variance [4].

We define a general superposition of two negative binomial states as follows

$$|\Psi_g\rangle = N_g[|w, q\rangle + \mu e^{i\psi}|w, q e^{i\phi}\rangle] \tag{3}$$

where μ is a real number and N_g is the normalization constant obtained by taking the norm of (3) and is given by

$$[N_g^2]^{-1} = 2\,\mathrm{Re}[(1-|q|^2)^{w+1}\{(1-|q|^2)^{-(w+1)} + \mu \mathrm{e}^{-\mathrm{i}\psi}(1-|q|^2\mathrm{e}^{\mathrm{i}\phi})^{-(w+1)}\}]. \tag{4}$$

For $\mu = 0$, $|\Psi_g\rangle \to |q, w\rangle$ and for $\mu = \infty$, $|\Psi_g\rangle \to |q\mathrm{e}^{\mathrm{i}\phi}, w\rangle$. The state (3) represents an even negative binomial state $|\Psi_e\rangle$ when $\mu = 1$, $\psi = 0$, $\phi = \pi$; an odd negative binomial state $|\Psi_o\rangle$ when $\mu = 1$, $\psi = \pi$, $\phi = \pi$; and an oblique negative binomial state $|\Psi_p\rangle$ when $\mu = 1$, $\psi = 0$, $\phi = \pi/2$. The normalization constants represented by N_e, N_o, and N_p for the even, odd and oblique negative binomial states respectively are given by the following expressions:

$$\begin{aligned}
[N_e^2]^{-1} &= 2(1-|q|^2)^{w+1}\{(1-|q|^2)^{-(w+1)} + (1+|q|^2)^{-(w+1)}\} \\
[N_o^2]^{-1} &= 2(1-|q|^2)^{w+1}\{(1-|q|^2)^{-(w+1)} - (1+|q|^2)^{-(w+1)}\} \\
[N_p^2]^{-1} &= 2\,\mathrm{Re}[(1-|q|^2)^{w+1}\{(1-|q|^2)^{-(w+1)} + (1-\mathrm{i}|q|^2)^{-(w+1)}\}].
\end{aligned} \tag{5}$$

It is quite straightforward to show that the states $|\Psi_e\rangle$, $|\Psi_o\rangle$, and $|\Psi_p\rangle$ reduce to even, odd and oblique coherent states respectively in the limiting condition of parameters: $w \to \infty$, $|q|^2 \to 0$, such that $(w+1)|q|^2/(1-|q|^2) \equiv \langle n\rangle$ is unchanged. In the other limiting condition, i.e. $w \to 0$, $|q|^2 > 0$ the states $|\Psi_e\rangle$, $|\Psi_o\rangle$ and $|\Psi_p\rangle$ reduce to the so-called even, odd and oblique thermal (pure) states. We shall now make use of the states $|\Psi_e\rangle$, $|\Psi_o\rangle$ and $|\Psi_p\rangle$ to calculate the mean photon number of these states. The mean photon number is defined as the expectation value of the number operator $\hat{n} = a^+a$. It is easy to show that the expression of the mean photon number $\langle n\rangle_e$, $\langle n\rangle_o$ and $\langle n\rangle_p$ for the even, odd and oblique negative binomial states are

$$\begin{aligned}
\langle n\rangle_e &= \frac{(w+1)|q|^2[(1-|q|^2)^{-(w+2)} - (1+|q|^2)^{-(w+2)}]}{[(1-|q|^2)^{-(w+1)} + (1+|q|^2)^{-(w+1)}]} \\
\langle n\rangle_o &= \frac{(w+1)|q|^2[(1-|q|^2)^{-(w+2)} + (1+|q|^2)^{-(w+2)}]}{[(1-|q|^2)^{-(w+1)} - (1+|q|^2)^{-(w+1)}]} \\
\langle n\rangle_p &= \frac{(w+1)|q|^2\,\mathrm{Re}[(1-|q|^2)^{-(w+2)} + \mathrm{i}(1-\mathrm{i}|q|^2)^{-(w+2)}]}{\mathrm{Re}[(1-|q|^2)^{-(w+1)} + (1-\mathrm{i}|q|^2)^{-(w+1)}]}.
\end{aligned} \tag{6}$$

Again, we obtain $\langle n\rangle_e = \langle n\rangle \tanh(\langle n\rangle)$, $\langle n\rangle_o = \langle n\rangle \coth(\langle n\rangle)$ in the coherent state limit, which as a matter of fact are precisely the expressions of mean photon numbers for the even and the odd coherent states respectively [12].

The second-order zero-time coherence function defined as

$$g^{(2)}(0) = \frac{\langle a^{+2}a^2\rangle}{\langle a^+a\rangle^2} \tag{7}$$

is an important quantity which can provide a measure of statistical nature of any electromagnetic field state. $g^{(2)}(0) < 1$ implies the sub-Poissonian statistics. In the following we give the expressions of $g_e^{(2)}(0)$, $g_o^{(2)}(0)$, and $g_p^{(2)}(0)$, i.e. the second-order coherence functions for the even, odd and oblique negative binomial states respectively:

$$g_e^{(2)}(0) = \frac{(w+2)}{(w+1)}\left[1 + \frac{(1-|q|^4)^{-(w+1)}\{(1+|q|^2)^{-1} + (1-|q|^2)^{-1}\}^2}{[(1-|q|^2)^{-(w+2)} - (1+|q|^2)^{-(w+2)}]^2}\right] \tag{8a}$$

$$g_o^{(2)}(0) = \frac{(w+2)}{(w+1)}\left[1 - \frac{(1-|q|^4)^{-(w+1)}\{(1+|q|^2)^{-1} - (1-|q|^2)^{-1}\}^2}{[(1-|q|^2)^{-(w+2)} + (1+|q|^2)^{-(w+2)}]^2}\right] \tag{8b}$$

$$g_p^{(2)}(0) = \frac{\mathrm{Re}\{(w+1)(w+2)|q|^4[(1-|q|^2)^{-(w+3)} - (1-\mathrm{i}|q|^2)^{-(w+3)}]\}}{2(1-|q|^2)^{w+1}N_p^2\langle n\rangle_p^2}. \tag{8c}$$

In the coherent state limit of the parameters w and $|q|^2$, we can recover the expression of second-order coherence function of even, odd and oblique coherent states from (8). In fact, in this limit we obtain from (8), $g_e^{(2)}(0) = \coth^2(\langle n\rangle)$ and $g_o^{(2)}(0) = \tanh^2(\langle n\rangle)$ which are exactly (63) and (64) of [12].

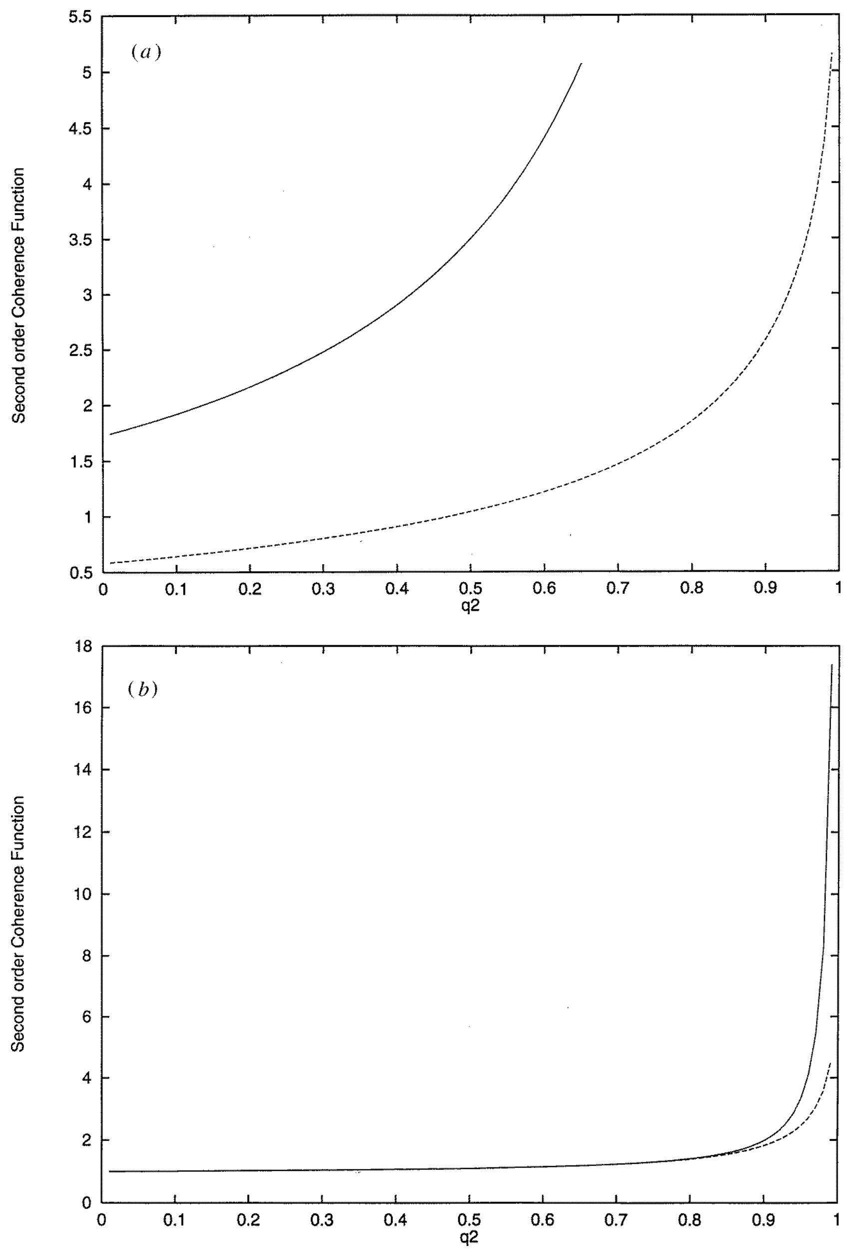


Figure 1. The second-order coherence function $g^{(2)}(0)$ as a function of parameter $q2 = |q|^2$ with $q = |q|$ and (*a*) $\langle n\rangle = 1$, (*b*) $\langle n\rangle = 10$. The full curve shows the even state, and the broken curve shows the odd state with $\langle n\rangle \equiv (w+1)q2/(1-q2)$.

We plot the second-order coherence function (8) as a function of parameter $q2 = |q|^2 (0 \leqslant q2 < 1)$ for even and odd negative binomial states in figures 1(a) and 1(b) for two different values of $\langle n\rangle$. For the lower value of $\langle n\rangle$, the odd state exhibits sub-Poissonian behaviour (figures 1(a)) for most of the values of $q2$, but, the even state always shows super-Poissonian behaviour for the complete range of $q2$ irrespective of the value of $\langle n\rangle$ as could be noted from formula (8a). With increases in $\langle n\rangle$ the difference between the values of $g_e^{(2)}(0)$ and $g_o^{(2)}(0)$ reduces considerably, especially at low values of $q2$ and their value remains around 1, which in fact is very clear from figure 1(b). Physically, it means that the non-classical nature of at least one of these states is apparent when the photon occupation number is very small. Nevertheless, the second order coherence function for these states start behaving like ordinary coherent states for the high photon occupation number $\langle n\rangle$ (up to a certain range of the parameter $q2$). We have also verified the nature of $g_p^{(2)}(0)$ as a function of $q2$ and it does not show any sub-Poissonian behaviour irrespective of any mean photon number ($\langle n\rangle$) values.

Before proceeding further we would like to give operator representation of the even and odd negative binomial states in terms of the so-called even and odd thermal states, e.g.

$$\rho_{\text{even}}^{nb} = \frac{4N_e^2}{w_e!}\frac{(1-|q|^2)^{(w_e+1)}}{(1-|q|^4)}(|q|^2)^{-w_e}(a)^{w_e}(\rho_{\text{even}}^{th})(a^+)^{w_e} \tag{9}$$

when $w \equiv w_e$ is even;

$$\rho_{\text{even}}^{nb} = \frac{4N_e^2}{w_o!}\frac{(1-|q|^2)^{(w_o+1)}}{(1-|q|^4)}(|q|^2)^{-(w_0-1)}(a)^{w_o}(\rho_{\text{odd}}^{th})(a^+)^{w_o} \tag{10}$$

when $w \equiv w_o$ is odd;

$$\rho_{\text{odd}}^{nb} = \frac{4N_o^2}{w_e!}\frac{(1-|q|^2)^{(w_e+1)}}{(1-|q|^4)}(|q|^2)^{-(w_e-1)}(a)^{w_e}(\rho_{\text{odd}}^{th})(a^+)^{w_e} \tag{11}$$

when $w \equiv w_e$ is even;

$$\rho_{\text{odd}}^{nb} = \frac{4N_o^2}{w_o!}\frac{(1-|q|^2)^{(w_o+1)}}{(1-|q|^4)}(|q|^2)^{-w_o}(a)^{w_o}(\rho_{\text{even}}^{th})(a^+)^{w_o} \tag{12}$$

when $w \equiv w_o$ is odd; where $\rho_{\text{even}}^{nb}(\rho_{\text{odd}}^{nb})$ and $\rho_{\text{even}}^{th}(\rho_{\text{odd}}^{th})$ are the density operators for the even (odd) negative binomial states and even (odd) thermal states respectively. For example, the expression of ρ_{even}^{th} reads as

$$\rho_{\text{even}}^{th} = \sum_{n=0}^{\infty}[1-(q2)^2](q2)^{2n}|2n\rangle\langle 2n|. \tag{13}$$

In (9)–(12), we have taken only the diagonal terms in the density matrix into account to make their connection to the thermal density of the non-pure state.

3. Squeezing

In this section we study the squeezing properties of the even and odd negative binomial states. It is well known that the quadrature operators of the single mode field are given by

$$X_1 = \tfrac{1}{2}(a+a^+) \qquad X_2 = \frac{1}{2i}(a-a^+) \tag{14}$$

such that $[X_1, X_2] = \frac{i}{2}$ which implies the uncertainty relation

$$(\Delta X_1)^2(\Delta X_2)^2 \geqslant \tfrac{1}{16} \tag{15}$$

where the variance $(\Delta X_i)^2 = \langle X_i^2\rangle - \langle X_i\rangle^2$. The field is said to be squeezed if $(\Delta X_i)^2 < \frac{1}{4}$ $(i = 1, 2)$. In terms of the normally ordered variances $:(\Delta X_i)^2: = \langle :X_i^2: \rangle - \langle X_i\rangle^2$ and the squeezing exits when $:(\Delta X_i)^2: < 0$. Note that for both even and odd states under consideration in this work we always find $\langle X_i\rangle = 0$ so that

$$S_1 = :(\Delta X_1)^2: = (\Delta X_1)^2 - \tfrac{1}{4} = \tfrac{1}{4}\langle 2a^+a + a^{+2} + a^2\rangle \tag{16}$$

$$S_2 = :(\Delta X_2)^2: = (\Delta X_2)^2 - \tfrac{1}{4} = \tfrac{1}{4}\langle 2a^+a - a^{+2} - a^2\rangle. \tag{17}$$

However, for the oblique state we find $\langle X_i\rangle \neq 0$, so the expressions (16) and (17) will be modified for that state. In order to calculate normal squeezing (S_1) as discussed above we need to know the expectation values of the even powers of the operators $a^+(a)$ for both even and odd negative binomial states. These expectation values are of the form

$$\langle\Psi_e|a^{+2s}|\Psi_e\rangle = \frac{(1-|q|^2)^{(w+1)}}{N_e^2}\sum_n \frac{[(n+2s+w)]!(n+w)!]^{1/2}}{n!w!}|q|^{2n}q^{*2s}[1+(-1)^n] \tag{18}$$

for the even state,

$$\langle\Psi_o|a^{+2s}|\Psi_o\rangle = \frac{(1-|q|^2)^{(w+1)}}{N_o^2}\sum_n \frac{[(n+2s+w]!(n+w)!]^{1/2}}{n!w!}|q|^{2n}q^{*2s}[1-(-1)^n] \tag{19}$$

for the odd state.

Similarly, the expectation values of $\langle a^{2s}\rangle$ can be calculated. We make use of these expressions together with the equations (16) and (17) to calculate the in phase squeezing S_1 for both even and odd negative binomial states. We define $q = |q|\mathrm{e}^{\mathrm{i}\alpha}$ and observe no squeezing for $\alpha = 0$. Next, we set $\alpha = \pi/2$ and plot S_1 as a function of parameter $q2 = |q|^2$ in figures 2(a) and 2(b) for $\langle n\rangle = 2$ and 4 respectively. From these figures it is quite clear that both even and odd negative binomial states do exhibit squeezing, however, the depth of squeezing and the range of $q2$ over which squeezing is observed are very sensitive to the value of $\langle n\rangle$ and they decrease with the increase of $\langle n\rangle$ for the even state, for example. Incidentally, similar behaviour has also been observed in even and odd coherent states. Also, as $\langle n\rangle$ increases, the disparity of the squeezing values of even and odd states (as a function of $q2$) up to $q2 = 0.6$ or so diminishes (figure 2(b)).

4. Wigner function

In this section we study the Wigner function for even, odd and oblique negative binomial states. With the study of the Wigner function one can characterize the non-classical nature of the field states.

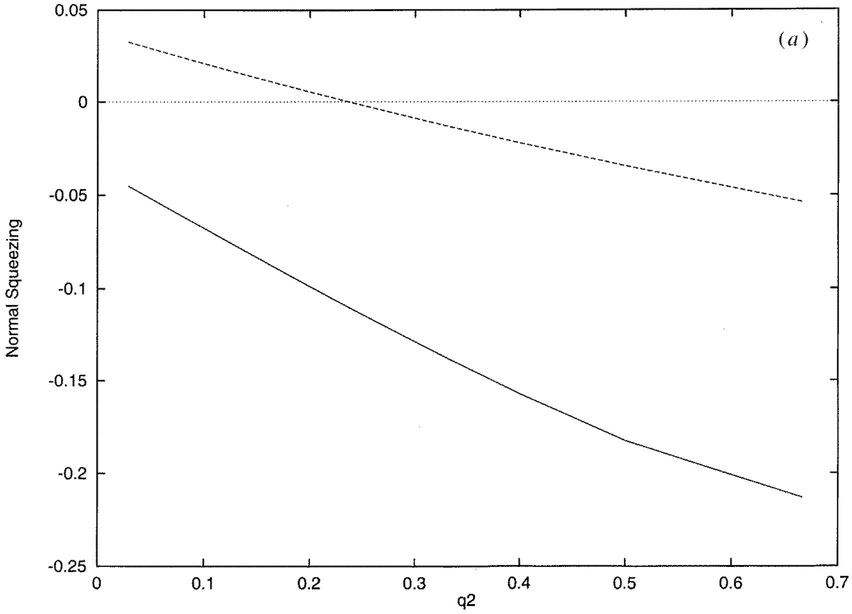
The Wigner function $W(\alpha)$ is defined [17] in terms of the characteristic function $C_p(\beta)$ by

$$W(\alpha) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \mathrm{d}^2\beta\, C_p(\beta)\exp(\alpha\beta^* - \beta\alpha^*)\exp(-\tfrac{1}{2}|\beta|^2). \tag{20}$$

The characteristic function is defined as the expectation value of the Glauber translation operator

$$C_p(\beta) \equiv \mathrm{Tr}[\hat\rho\exp(\beta a^+)\exp(-\beta^* a)] \tag{21}$$

where $\hat\rho$ is the density matrix operator: $\rho_e = |\Psi_e\rangle\langle\Psi_e|$, $\rho_o = |\Psi_o\rangle\langle\Psi_o|$, and $\rho_p = |\Psi_p\rangle\langle\Psi_p|$ for even, odd and oblique states respectively.



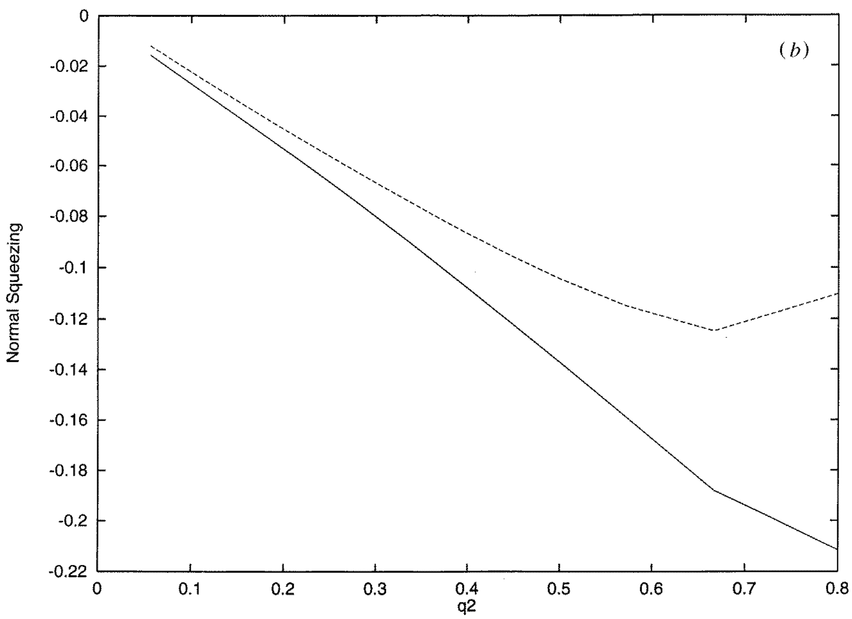


Figure 2. The normal squeezing (in-phase squeezing) $S_1 \equiv [(\Delta X_1)^2 - \frac{1}{4}]$ as a function of $q2 = |q|^2$ with $q = |q|e^{i\pi/2}$ and (*a*) $\langle n\rangle = 2$, (*b*) $\langle n\rangle = 4$. The full curve shows the even state, and the broken curve shows the odd state with $\langle n\rangle \equiv (w+1)q2/(1-q2)$.

Substituting (21) in (20) and performing the integration give the following expressions for the Wigner function

$$W(\alpha)|_{\text{even}} = \frac{8}{\pi} N_e^2 (1-|q|^2)^{(w+1)} \sum_n \frac{(2n+w)!}{(2n)!w!} (|q|^2)^{2n} e^{-2|\alpha|^2} L_{2n}(4|\alpha|^2) \tag{22a}$$

$$W(\alpha)|_{\rm odd} = \frac{8}{\pi} N_o^2 (1-|q|^2)^{(w+1)} \sum_n \frac{(2n+w+1)!}{(2n+1)!w!} (-|q|^2)^{2n+1} {\rm e}^{-2|\alpha|^2} L_{2n+1}(4|\alpha|^2) \qquad (22b)$$

$$W(\alpha)|_{\rm oblique} = \frac{4}{\pi} N_p^2 (1-|q|^2)^{(w+1)} \sum_n \frac{(n+w)!}{n!w!} (-|q|^2)^n {\rm e}^{-2|\alpha|^2} [1+\cos(n\pi/2)] L_n(4|\alpha|^2). \qquad (22c)$$

In obtaining the expressions of the Wigner function (as above) we have assumed phase averaging such that only the diagonal terms of the density operator take part in determining these expressions. In other words we are examining a special case (known as mixed states in the literature because of phase averaging) of the pure states defined above. Note that by invoking phase averaging we are able to establish relationships between the density operators of even (odd) negative binomial states and even (odd) thermal states (see (9)–(12) above). What is its equivalent for the Wigner function has precisely motivated us to carry out the study of the phased averaged Wigner function. As we will see such phased averaged Wigner function is a special case of the more generalized Wigner function (see (24) below) and interpolates between the Wigner function of the even (odd) thermal state and the phased averaged even (odd) coherent state. We will also see that the phase averaged Wigner function can be put in a closed form.

The Wigner functions for even, odd and oblique negative binomial states are plotted for different values of parameters w and $q2(=|q|^2)$ (keeping $\langle n\rangle = (w+1)q2/(1-q2)$ unchanged) in figures 3, 4 and 5 respectively. The Wigner function of the even (odd, oblique) negative binomial state reduces to the Wigner function of an even (odd, oblique) coherent state or an even (odd, oblique) thermal state under the two different limiting conditions of parameters as depicted in these figures. The Wigner function of the even negative binomial state starts with a central peak with a crater-like structure around it (figure 3(a)) near the coherent limit. The depth of the crater reduces considerably as one moves towards the thermal limit (figures 3(b) and (c)). The situation for the odd negative binomial state looks inverted in comparison to the even negative binomial state (note the minus sign appearing outside the formula (22b)), because the Wigner function of this state starts with an inverted peak surrounded by an inverted crater (figure 4(a)) near its coherent limit and the crater height reduces as one moves towards the thermal limit (figure 4(b) and (c)). The evolution of the Wigner function for the oblique negative binomial state is very interesting. It starts with a doubly folded inverted peak near the coherent state limit (figure 5(a)); as the parameters are changed towards the thermal limit, one of them starts to unfold towards the positive side (figure 5(b)) and this unfolds completely in the thermal limit (figure 5(c)) and consequently the one-fold inverted peak structure remains. Since we have used phase averaging of the density operator in deriving the expressions of the Wigner function so that they can conveniently be put in a closed form:

$$W(\alpha)|_{\rm even} = \frac{4}{\pi} N_e^2 (1-|q|^2)^{(w+1)} {\rm e}^{-2|\alpha|^2} \bigg[(1+|q|^2)^{-(w+1)}\, {}_1F_1\left(w+1, 1, \frac{4|\alpha|^2|q|^2}{|q|^2+1}\right) + (1-|q|^2)^{-(w+1)}\, {}_1F_1\left(w+1, 1, \frac{4|\alpha|^2|q|^2}{|q|^2-1}\right) \bigg] \qquad (23a)$$

$$W(\alpha)|_{\rm odd} = \frac{4}{\pi} N_o^2 (1-|q|^2)^{(w+1)} {\rm e}^{-2|\alpha|^2} \bigg[(1+|q|^2)^{-(w+1)}\, {}_1F_1\left(w+1, 1, \frac{4|\alpha|^2|q|^2}{|q|^2+1}\right) - (1-|q|^2)^{-(w+1)}\, {}_1F_1\left(w+1, 1, \frac{4|\alpha|^2|q|^2}{|q|^2-1}\right) \bigg] \qquad (23b)$$

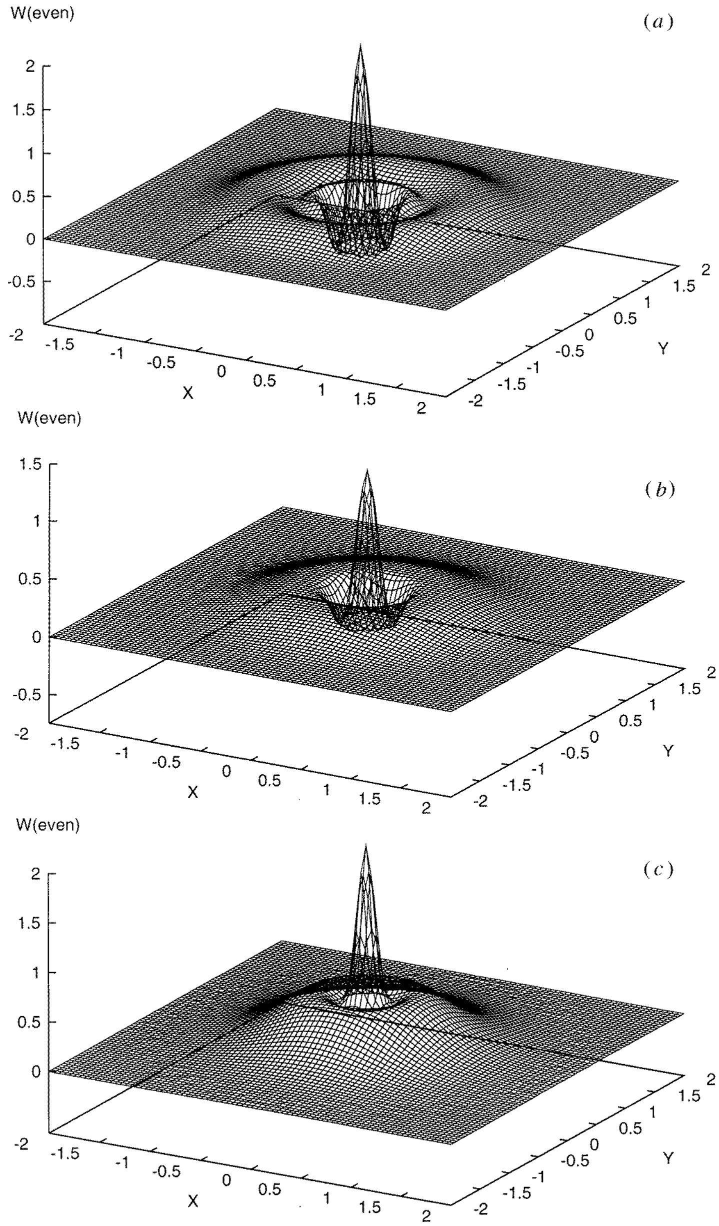


Figure 3. Wigner function $W(X, Y)$ for the even negative binomial state with $\langle n\rangle = 4$, $X = \mathrm{Re}(\alpha)$ and $Y = \mathrm{Im}(\alpha)$. (*a*) $w = 75$, $q2 = 0.05$ (coherent state limit), (*b*) $w = 5$, $q2 = 0.4$ and (*c*) $w = 0$, $q2 = 0.8$ (thermal state limit).

$$W(\alpha)|_{\mathrm{oblique}} = \frac{4}{\pi} N_e^2 (1 - |q|^2)^{(w+1)} \mathrm{e}^{-2|\alpha|^2} \left[(1 + |q|^2)^{-(w+1)} \,{}_1F_1 \left(w + 1, 1, \frac{4|\alpha|^2|q|^2}{|q|^2 + 1} \right) + \mathrm{Re}(1 + \mathrm{i}|q|^2)^{-(w+1)} \,{}_1F_1 \left(w + 1, 1, \frac{4\mathrm{i}|\alpha|^2|q|^2}{\mathrm{i}|q|^2 + 1} \right) \right] \qquad (23c)$$

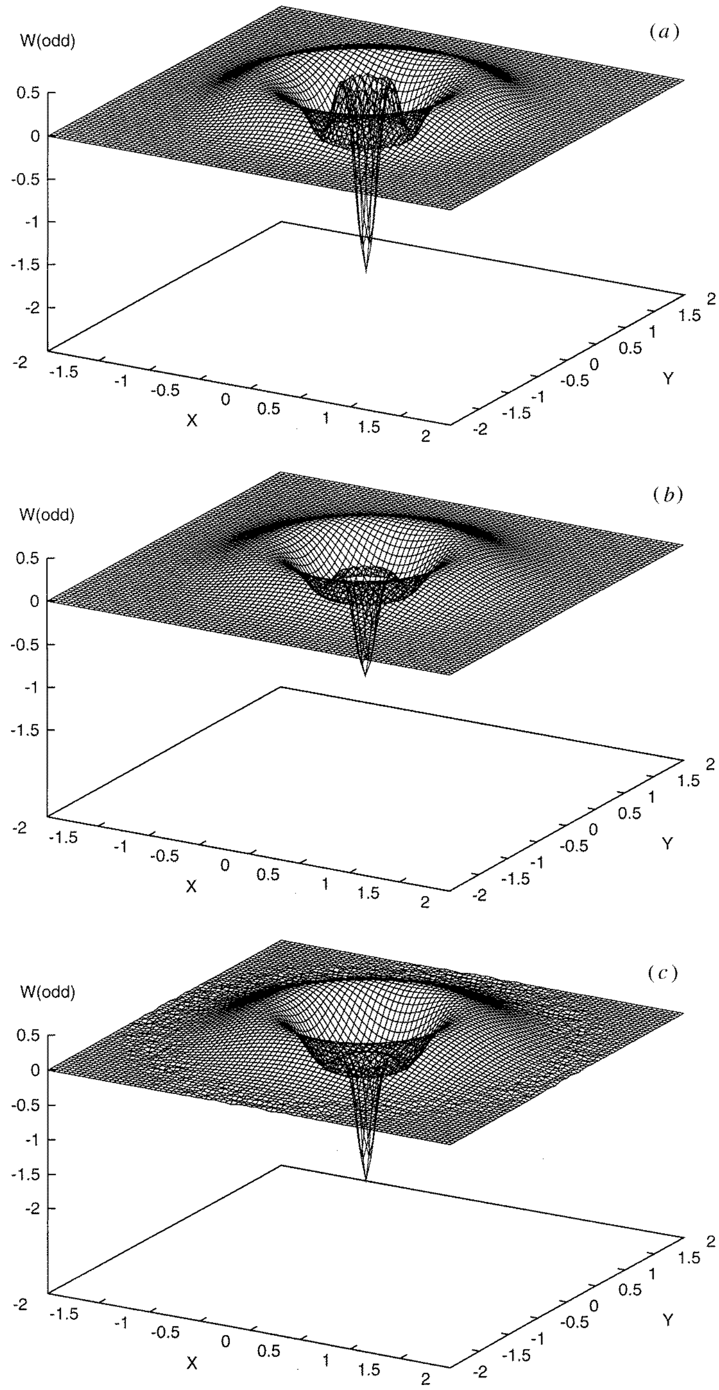


Figure 4. Wigner function $W(X, Y)$ for the odd negative binomial state with all other parameters as in figure 3.

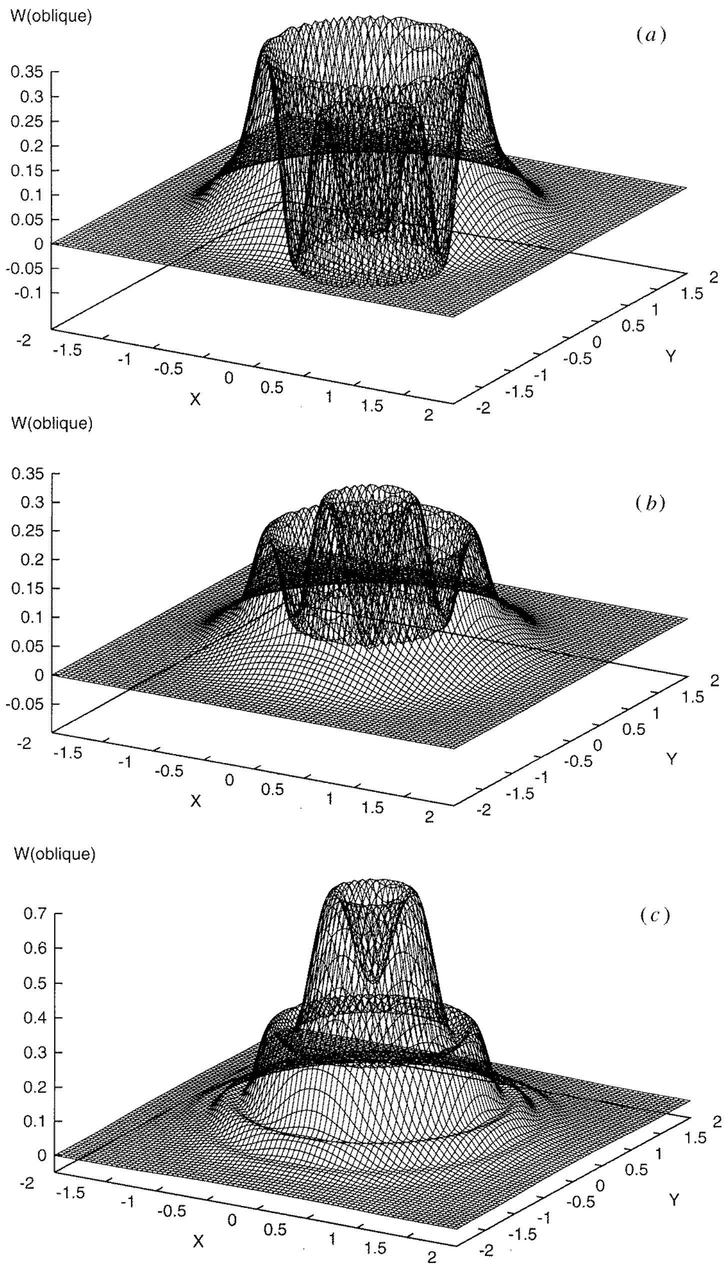


Figure 5. Wigner function $W(X, Y)$ for the oblique negative binomial state with all other parameters as in figure 3.

where ${}_1F_1(a, b; z)$ is the confluent hypergeometric function:

$$ {}_1F_1(a, b; z) = \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \frac{z^r}{r!}. $$

When we take the complete density matrix for even, odd and oblique negative binomial

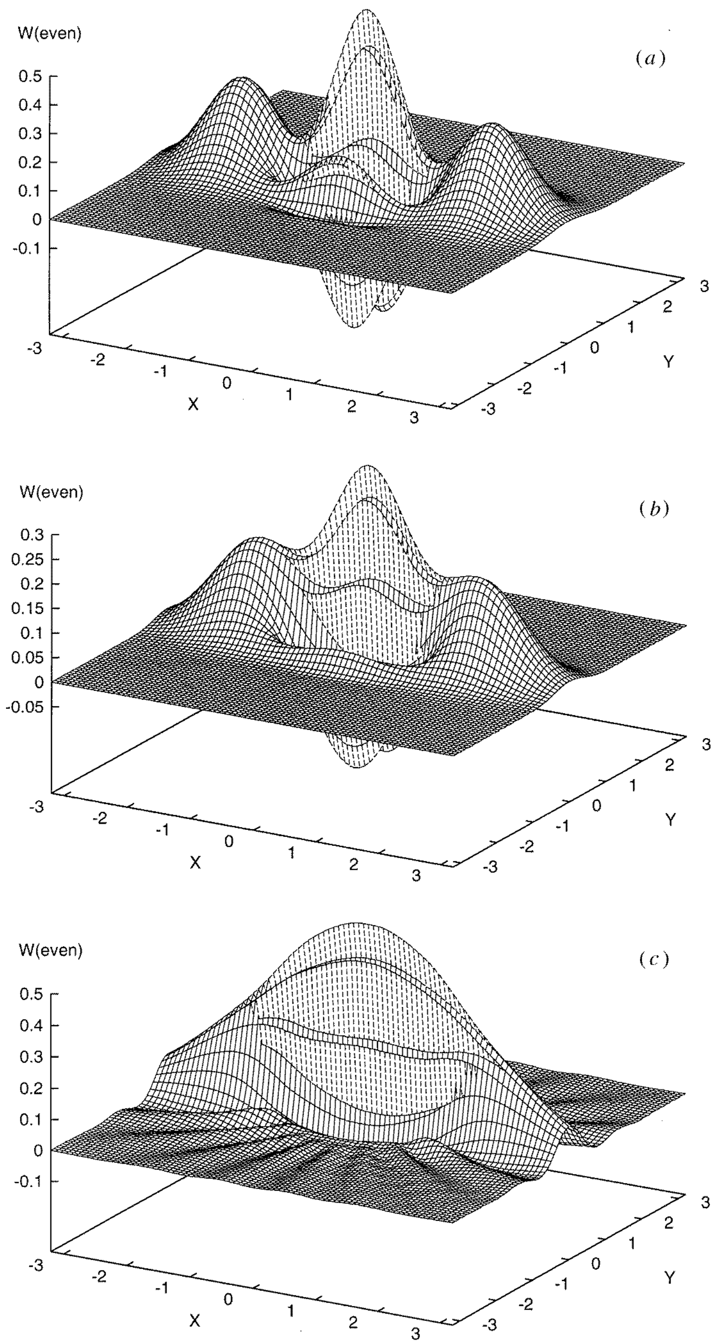


Figure 6. The complete Wigner function $W(X, Y)$ (as defined by (24)) for the even negative binomial state with $\langle n \rangle = 4$, $X = \text{Re}(\alpha)$ and $Y = \text{Im}(\alpha)$. (*a*) $w = 75$, $q2 = 0.05$ (coherent state limit), (*b*) $w = 5$, $q2 = 0.4$ and (*c*) $w = 0$, $q2 = 0.8$ (thermal state limit).

states in calculating the Wigner function, we have to take into account the non-diagonal terms also. In that case the expression of the Wigner function take the following form (after

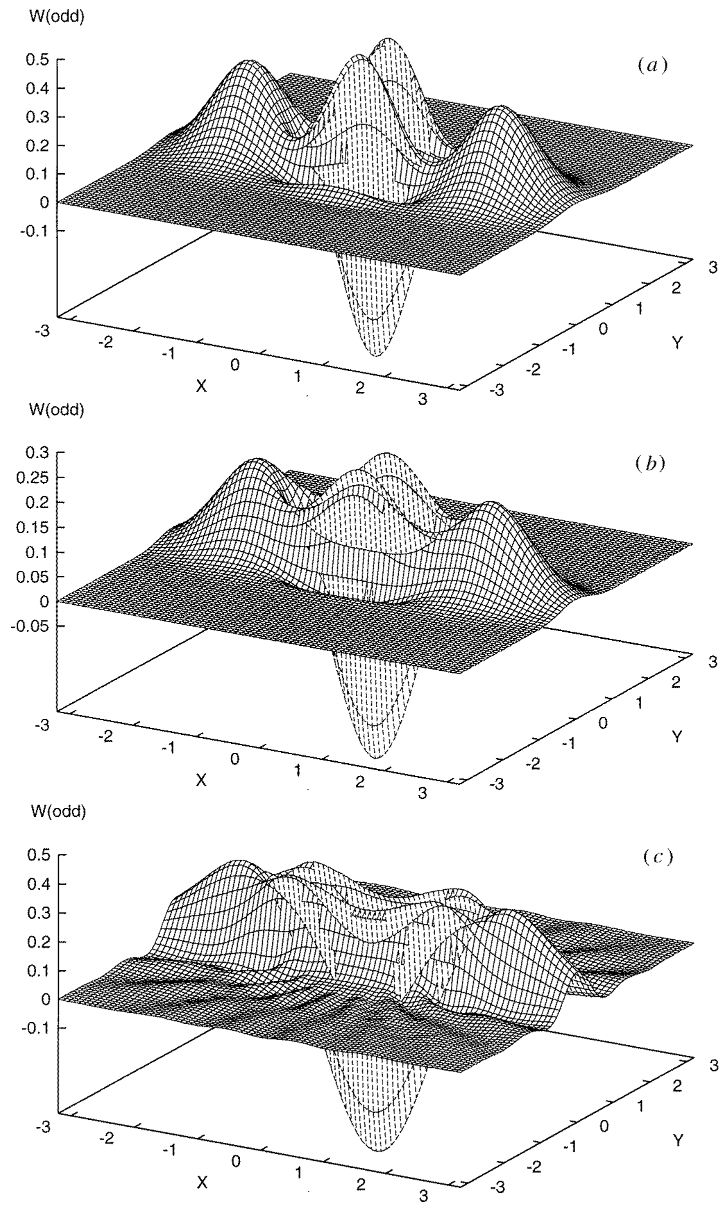


Figure 7. The complete Wigner function $W(X, Y)$ (as defined by (24)) for the odd negative binomial state with all other parameters as in figure 6.

some simplifications):

$$W(\alpha)|_q = \frac{1}{\pi}\Bigg[\sum_n 2(-1)^n \mathrm{e}^{-2|\alpha|^2} L_n(4|\alpha|^2)\rho_q(n,n) + \sum_{m=1}\sum_n \sqrt{\{n!/(m+n)!\}}2(-1)^n \mathrm{e}^{-2|\alpha|^2} L_n^m(4|\alpha|^2) \times\{[2\alpha^*]^m \rho_q(m+n,n) + [2\alpha]^m \rho_q(n,n+m)\}\Bigg] \quad (24)$$

where the subscript q stands for e, o and p for even, odd and oblique negative binomial states respectively. $\rho_q(n, n)$ is the diagonal element of the density operator while $\rho_q(m+n, n)$ or $\rho_q(n, m+n)$ represents the off-diagonal element of the density operator. The definition of the density operator ρ_e, ρ_o, ρ_p for even, odd and oblique negative binomial states respectively has been mentioned just after (21). Thus by inserting the appropriate density operator of the state (ρ_e, ρ_o or ρ_p) in (24) one can obtain the corresponding Wigner function (W_e, W_o or W_p) of that state. In (24) $L_n^\alpha(x)$ is the associated Laguerre polynomial:

$$L_n^\alpha(x) = \sum_{j=1}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}. \tag{25}$$

Figures 6(a) and 7(a) show the Wigner function (as defined by (24)) representing the even and the odd coherent state ($\bar{n} = 4$, $w = 75$, $q2 = 0.05$) in phase space. This distribution is a real function which can take negative values in limited regions of phase space. The two Gaussian peaks are associated to the $|\alpha\rangle$ and $|-\alpha\rangle$ ($\alpha = 2$) coherent fields and the oscillatory part in between is directly related to the coherence between them. The existence of the oscillations associated with the negative quasi-probabilities values, is a signature of the non-classical state. Such a state is very different from a statistical mixture involving the same coherent field parts for which the Wigner distribution merely exhibits two separated peaks without superimposed oscillations. In this case of even and odd coherent states the two Gaussian peaks are well separated and compact. Note that the oscillatory part is very much different from each other in two cases. The width of the Gaussian peak is associated with the variance of the coherent field $\sim \sqrt{(\bar{n})}$. If we move over to a typical even (odd) negative binomial state with $\bar{n} = 4$ ($w = 5$, $q2 = 0.4$), the Wigner function still shows two Gaussian-like peaks (figures 6(b) and 7(b)) and the oscillations in between these peaks. However, the widths of the so-called Gaussian-like peaks are much broader as compared to the coherent state case. This could be because the variance of the negative binomial state ($\sim \bar{n}/(1-q2)$) is more (super-Poissonian) than the coherent state. The interference between the two negative binomial states is modified considerably as compared with the coherent state case because of the change in the statistics or photon number distribution of these states. There is a reduction in the non-classical character of the even state because it is becoming less negative in its quasi-probability distribution. Hence the coherence or quantum interferences are sensitive to the statistics of the constituent states of the 'Schrödinger cat'. In other words the quasi-probability for the even (odd) negative binomial state is both qualitatively, as well as quantitatively, different from the even (odd) coherent state.

Finally, we move to the other extreme of the negative binomial state known as the quasi-thermal state (pure state) by setting parameters $w = 0$, $q2 = 0.8$, $\bar{n} = 4$. The Wigner function for the even and odd quasi-thermal states are depicted in figures 6(c) and 7(c) respectively. Clearly the Gaussian-like peaks have undergone a considerable amount of broadening such that we see only one prominent peak (positive in the even state and negative in the odd state) in the Wigner function of these states along with a few oscillations superimposed on them. Note that the non-classical character of the even quasi-thermal state is reduced considerably in comparison with the odd quasi-thermal state. Thus as we move from the coherent 'Schrödinger cat' to a quasi-thermal 'Schrödinger cat' we find a reduced non-classical character in the even state and a reduced 'coherence' (i.e. less oscillatory behaviour) in general for both the even and odd states.

The Wigner function (24) for the oblique negative binomial state ($\psi = 0, \phi = \pi/2$) is equally interesting. In the coherent limit (figure 8(a), $w = 75$, $q2 = 0.05$, $\bar{n} = 4$) we observe three Gaussian-like peaks in the quasi-probability distribution. There are two small but equal peaks (oblique peaks) symmetrically located opposite to each other. The third

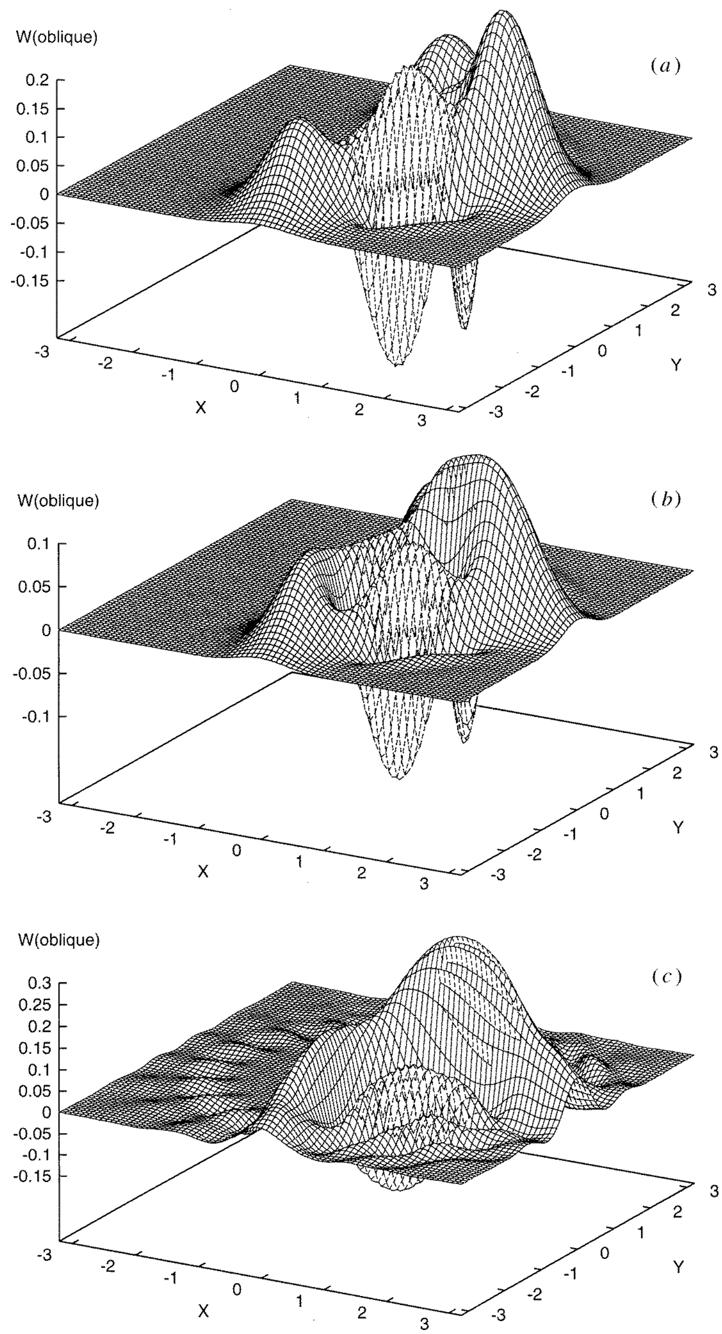


Figure 8. The complete Wigner function $W(X, Y)$ (as defined by (24)) for the oblique negative binomial state with all other parameters as in figure 6.

peak is bigger in height and is located at right angles to the line joining the small peaks. In between the small and big peaks we observe oscillations which are manifestations of the coherence between the oblique states. For a typically negative binomial state (figure 8(*b*),

$w = 5$, $q2 = 0.4$, $\bar{n} = 4$) all the peaks become broader and the oscillations in between them are reducing. Finally in the quasi-thermal limit (figure 8(c), $w = 0$, $q2 = 0.8$, $\bar{n} = 4$) the third peak has overtaken the other two peaks in height as well as in width and the oscillations are becoming less and less prominent. Clearly, here also (as in the even state) the changes in the statistics from Poissonian to super-Poissonian in the component states taking part in the 'Schrödinger cat'-like superposition brings a reduction in the non-classical character as well as coherence between them.

5. Possibilities of generating even (odd) negative binomial states and conclusions

We next show the possibility of generating even and odd negative binomial states of the field. For that we first consider the production of an ordinary negative binomial state in the process of parametric amplification by a proper choice of the initial conditions. We consider $SU(1,1)$ coherent states for this purpose. These states are defined by [18]

$$|\zeta\rangle = (1-|\zeta|^2)^k \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)}\right]^{1/2} \zeta^n |k, n+k\rangle. \tag{26}$$

$|k, n+k\rangle \equiv |n\rangle|n+k\rangle$ represents a two-mode Fock state, i.e. eigenstate of the operator $\hat{a}_1\hat{a}_2$, where $\hat{a}_1$ and $\hat{a}_2$ represent annihilation operators of the two different modes of the electromagnetic field.

The diagonal elements of (26) have distributions given by

$$P(n) = (1-|\zeta|^2)^{2k} \sum_{n=0}^{\infty} \frac{\Gamma(n+2k)}{\Gamma(n+1)\Gamma(2k)} |\zeta|^{2n} \tag{27}$$

which is the same as the negative binomial distribution with $s = 2k - 1$. The $SU(1,1)$ algebra can be realized in terms of two modes a and b of the field, i.e. $K_+ = a^+b^+$, $K_- = ab$, $K_3 = \frac{1}{2}(a^+a + b^+b + 1)$ etc. So, in terms of the Fock states $|n, m\rangle$ of the two-mode radiation field the parameter k equals $(m - n)$ and (26) can be written as

$$|\zeta\rangle = (1-|\zeta|^2)^{(1+s)/2} \sum_{n=0}^{\infty} \left[\frac{\Gamma(n+s+1)}{\Gamma(n+1)\Gamma(s+1)}\right]^{1/2} \zeta^n |n+s, s\rangle \tag{28}$$

or

$$|\zeta\rangle = \exp[\gamma(a^+b^+ - ab)] \frac{a^{+s}}{\sqrt{s!}} |0,0\rangle \qquad \zeta = \tanh(\gamma). \tag{29}$$

So the state $|\zeta\rangle$ is essentially the negative binomial state of a two-mode radiation field in which the probability of finding n signal photons obeys the negative binomial distribution. The interaction Hamiltonian of parametric amplification is $\mathrm{i}(a^+b^+ - ab)$ and it would produce the state (29) provided the input to the amplifier is such that the difference between the idler and signal photon is s.

The so-obtained negative binomial state can be used to produce the even and odd negative binomial state by the method of Yurke and Stoler [13] employed for producing superposition of coherent states. For this purpose, we have to allow our ordinary negative binomial state field to propagate through an amplitude dispersive medium. Under suitable conditions we can obtain a quantum superposition of two negative binomial states.

Another method of generating the even (odd) negative binomial state is by allowing the ordinary negative binomial state to interact with a Kerr-like medium in one arm of the Mech–Zehnder interferometer and a resonant two-level atom crossing one of the output ports of the same interferometers. This methodology was reported recently for the possibilities of generating even and odd coherent states [19]. Since the negative binomial state represents a coherent state under one of the limiting conditions of its parameters hence for certain values of parameters one can adopt this method for generating the even (odd) negative binomial states.

We have introduced even, odd and oblique negative binomial states and discussed some of their statistical properties in this work. Since any negative binomial state interpolates between a quasi-thermal (pure) state and a coherent state so the results obtained here may be useful in carrying out a systematic study of statistical properties as one moves from even (odd, oblique) quasi-thermal state to an even (odd, oblique) coherent state.

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